KVADRATIK FORMALAR VA ULARNI KANONIK KO'RINISHGA KELTIRISH HAQIDA

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Annotatsiya. Har qanday ko'rinishdagi kvadratik formani biror-bir xosmas chiziqli almashtirish orqali kanonik ko'rinishga keltirilishi mumkin. Bunda berilgan kvadratik formani kanonik ko'rinishga keltiriladigan almashtirish bir qiymatli emas, ya'ni har qanday kvadratik formani turli ko'rinishdagi almashtirishlar bilan kanonik ko'rinishga keltirilishi mumkin. Agar kvadratik formada o'zgaruvchilarning kvadrati ishtirok etmasa, u holda chiziqli almashtirish yordamida uni hech bo'lmaganda bitta o'zgaruvchining kvadrati qatnashgan kvadratik formaga keltirish mumkin bo'ladi. Ushbu maqolada kvadratik formalarni kanonik ko'rinishga keltirish va kanonik ko'rinishga keltiruvchi birorta almashtirishni topish usullarini keltirib o'tamiz.

Kalit so'zlar*:* kvadratik formalar, kanonik ko'rinish, to'la kvadratga ajratish, chiziqli almashtirish, xosmas almashtirish, ortogonal almashtirish, ortonormallash.

Ma'lumki, n ta $x_1, x_2, ..., x_n$ noma'lumlarning $f(x)$ kvadratik formasi deb, har bir hadi bu noma'lumlarning kvadrati yoki ikkita noma'lumning ko'paytmasidan iborat bo'lgan

$$f = \sum_{i=1}^{n}\left(\sum_{j=1}^{n}\left(a_{ij}x_i x_j\right)\right)$$

yig'indiga aytiladi.

Kvadratik formaning a_{ij} koeffisiyentlaridan foydalanib,

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \ddots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \quad (*)$$

kvadrat matritsani tuzish mumkin. Bu yerda A matritsaning barcha xarakteristik ildizlari haqiqiy bo'lishi uchun $a_{ij} = a_{ji}$ deb faraz qilinadi. A matritsaning rangi (*) ***kvadratik formaning rangi*** deyiladi. A matritsa aynimagan bo'lsa, (*) kvadratik forma xosmas deyiladi.

Kvadratik formaning koeffisiyentlari haqiqiy yoki kompleks sonlar bo'lishiga bog'liq holda kvadratik forma ***haqiqiy*** yoki ***kompleks*** deyiladi.

(*) ni matritsa formada quyidagacha yozish mumkin:

$$f = X^T \cdot A \cdot X \qquad (**)$$

bu yerda X va X^T o'zaro transponirlangan matritsalar bo'lib, $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$.

Ikki noma'lumli kvadratik forma

$$f = X^T \cdot A \cdot X = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

$$f = a_{11}x_1^2 + (a_{12} + a_{21})x_1x_2 + a_{22}x_2^2 = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2$$

ko'rinishda bo'ladi. Bunda $a_{12} = a_{21}$.

Uchta no'malumli kvadratik forma esa

$$f = X^T \cdot A \cdot X = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

$$f = a_{11}x_1^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3 + a_{22}x_2^2 + a_{33}x_3^2$$

ko'rinishda bo'ladi. Bunda $a_{12} = a_{21}$, $a_{13} = a_{31}$, $a_{23} = a_{32}$.

Bu xossalardan foydalanib, quyidagi teoremani keltiramiz:

Teorema. *A matritsali n noma'lumli kvadratik forma ustida Q matritsali chiziqli almashtirish bajarilgandan so'ng, u $Q^T \cdot A \cdot Q$ matritsali yangi n noma'lumli kvadratik formaga aylanadi*

1-misol. $f(x_1, x_2) = 2x_1^2 + 4x_1x_2 - 3x_2^2$ kvadratik forma ustida

$$\begin{cases} x_1 = 2y_1 - 3y_2 \\ x_2 = y_1 + y_2 \end{cases}$$

almashtirish bajarilgandan so'ng, hosil bo'lgan yangi kvadratik formani toping.

Yechish: Bu yerda kvadratik formaning matritsasi $A = \begin{pmatrix} 2 & 2 \\ 2 & -3 \end{pmatrix}$, chiziqli almashtirishning matritsasi esa $Q = \begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix}$ ko'rinishda bo'ladi. U holda teoremaga asosan

$$A^* = Q^T \cdot A \cdot Q = \begin{pmatrix} 2 & 1 \\ -3 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 2 \\ 2 & -3 \end{pmatrix} \cdot \begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 13 & -17 \\ -17 & 3 \end{pmatrix}$$

Bundan quyidagi kvadratik formani hosil qilamiz:

$$g(y_1, y_2) = 13y_1^2 - 34y_1y_2 + 3y_2^2$$

Natija: Chiziqli almashtirish bajarilgandan so'ng, kvadratik formaning rangi o'zgarmaydi.

Ta'rif. *Agar (*) kvadratik formada turli noma'lumlarning ko'paytmalari oldidagi barcha koeffisiyentlar nolga teng bo'lsa, u holda bu forma kvadratik formaning* ***kanonik ko'rinishi*** *deb ataladi.*

Shunday qilib, quyidagi

$$f(y_1, y_2, ..., y_n) = b_1y_1^2 + b_2y_2^2 + ... + b_ny_n^2$$

ifoda (*) formaning kanonik ko'rinishi deyiladi.

Shuni alohida ta'kidlash kerakki, kanonik ko'rinishda noldan farqli koeffisiyentlar soni (*) kvadratik formaning rangiga teng bo'lishi kerak.

Teorema. *Har qanday kvadratik forma biror xosmas chiziqli almashtirish orqali kanonik ko'rinishga keltirilishi mumkin.*

Bu teoremani matematik induksiya metodi yordamida isbotlash mumkin. Masalan,

$$f(x_1, x_2, x_3) = 2x_1x_2 - 6x_2x_3 + 2x_1x_3$$

kvadratik formani

a) $\begin{cases} x_1 = \frac{1}{2}t_1 + \frac{1}{2}t_2 + 3t_3 \\ x_2 = \frac{1}{2}t_1 - \frac{1}{2}t_2 - t_3 \\ x_3 = t_3 \end{cases}$ xosmas chiziqli almashtirish yordamida

$g_1(t_1, t_2, t_3) = \frac{1}{2}t_1^2 - \frac{1}{2}t_2^2 + 6t_3^2$ kanonik ko'rinishga keltirish mumkin;

b) $\begin{cases} x_1 = t_1 + 3t_2 + 2t_3 \\ x_2 = t_1 - t_2 - 2t_3 \\ x_3 = t_2 \end{cases}$ xosmas chiziqli almashtirish yordamida

$g_2(t_1, t_2, t_3) = 2t_1^2 + 6t_2^2 - 8t_3^2$ kanonik ko'rinishga keltirish mumkin.

Berilgan kvadratik forma keltiriladigan kanonik ko'rinish bir qiymatli aniqlangan emas, ya'ni har qanday kvadratik forma turli usullar bilan turli ko'rinishdagi kanonik ko'rinishga keltirilishi mumkin.

(*) krvadratik formani kanonik ko'rinishda yozish uchun A matritsaning xarakteristik ildizlarini, ya'ni $|A - \lambda E|$ ko'phadning ildizlarini topamiz. Bu ildizlar esa kanonik ko'rinishning koeffisiyentlari bo'ladi. Ba'zi hollarda faqat kanonik ko'rinishini emas, balki bu ko'rinishga keltiruvchi almashtirishni bilish kerak bo'lib qoladi. Buning uchun berilgan A simmetrik matritsani diagonal ko'rinishga keltiruvchi

Q ortogonal matritsani yoki uning teskari matritsasi Q^{-1} ni topish va A matritsaning λ_0 xarakteristik ildizlaridan foydalanib,

$$(A-\lambda_0 E)\cdot X=0$$

sistemaning fundamental yechimlarini ortonormallash kifoya.

2-misol. Quyidagi

$$f(x_1,\, x_2,\, x_3)=x_1^2-2x_2^2+x_3^2+4x_1x_2-8x_1x_3-4x_2x_3$$

kvadratik formani kanonik ko'rinishga keltiring va almashtirishlardan birini toping.

Yechish 1: dastlab xususiy usul, ya'ni kvadratik formani to'la kvadratga ajratish yo'li bilan kanonik ko'rinishga keltirishni va almashtirishlardan birini topishni ko'rib chiqamiz:

$$f(x_1,\, x_2,\, x_3)=x_1^2-2x_2^2+x_3^2+4x_1x_2-8x_1x_3-4x_2x_3=$$
$$=x_1^2+2x_1(2x_2-4x_3)+(2x_2-4x_3)^2-2x_2^2+x_3^2-4x_2x_3-(2x_2-4x_3)^2=$$
$$=(x_1+2x_2-4x_3)^2-2x_2^2+x_3^2-4x_2x_3-4x_2^2+16x_2x_3-16x_3^2=$$
$$=(x_1+2x_2-4x_3)^2-6x_2^2-15x_3^2+12x_2x_3=$$
$$=(x_1+2x_2-4x_3)^2-6(x_2^2-2x_2x_3+x_3^2)-9x_3^2=$$
$$=(x_1+2x_2-4x_3)^2-6(x_2-x_3)-9x_3^2=y_1^2-6y_2^2-9y_3^2$$

Bundan

$$\begin{cases} y_1=x_1+2x_2-4x_3 \\ y_2=x_2-x_3 \\ y_3=x_3 \end{cases} \quad\Rightarrow\quad \begin{cases} x_1=y_1-2y_2+2y_3 \\ x_2=y_2+y_3 \\ x_3=y_3 \end{cases}$$

almashtirish kelib chiqadi. Demak, berilgan kvadratik forma $\begin{cases} x_1=y_1-2y_2+2y_3 \\ x_2=y_2+y_3 \\ x_3=y_3 \end{cases}$ almashtirish orqali $g(y_1,\, y_2,\, y_3)=y_1^2-6y_2^2-9y_3^2$ kanonik ko'rinishga keladi.

Tekshirish:

a) o'niga qo'yish orqali

$$g(y_1,\, y_2,\, y_3)=(y_1-2y_2+2y_3)^2-2(y_2+y_3)^2+y_3^2+$$
$$+4(y_1-2y_2+2y_3)(y_2+y_3)-8(y_1-2y_2+2y_3)y_3-4(y_2+y_3)y_3=$$
$$=y_1^2+4y_2^2+4y_3^2-4y_1y_2+4y_1y_3-8y_2y_3-2y_2^2-4y_2y_3-2y_3^2+y_3^2+$$
$$+4y_1y_2+4y_1y_3-8y_2^2-8y_2y_3+8y_2y_3+8y_3^2-$$
$$-8y_1y_3+16y_2y_3-16y_3^2-4y_2y_3-4y_3^2=y_1^2-6y_2^2-9y_3^2$$

b) $A^*=Q^T\cdot A\cdot Q$ matritsa tuzish orqali

$$A=\begin{pmatrix}1 & 2 & -4\\ 2 & -2 & -2\\ -4 & -2 & 1\end{pmatrix},\quad Q=\begin{pmatrix}1 & -2 & 2\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix},\quad Q^T=\begin{pmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ 2 & 1 & 1\end{pmatrix}$$

$$A^*=\begin{pmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ 2 & 1 & 1\end{pmatrix}\cdot\begin{pmatrix}1 & 2 & -4\\ 2 & -2 & -2\\ -4 & -2 & 1\end{pmatrix}\cdot\begin{pmatrix}1 & -2 & 2\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix}=$$

$$=\begin{pmatrix}1 & 2 & -4\\ 0 & -6 & 6\\ 0 & 0 & -9\end{pmatrix}\cdot\begin{pmatrix}1 & -2 & 2\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 0 & -6 & 0\\ 0 & 0 & -9\end{pmatrix}$$

Demak,

$$g(y_1,\,y_2,\,y_3)=(y_1,\,y_2,\,y_3)\cdot\begin{pmatrix}1 & 0 & 0\\ 0 & -6 & 0\\ 0 & 0 & -9\end{pmatrix}\cdot\begin{pmatrix}y_1\\ y_2\\ y_3\end{pmatrix}=y_1^2-6y_2^2-9y_3^2.$$

Yechish 2: endi xos son va xos vektorlar orqali kvadratik formani kanonik ko‘rinishga keltirishni va almashtirishlardan birini topishni ko‘rib chiqamiz:

$$A=\begin{pmatrix}1 & 2 & -4\\ 2 & -2 & -2\\ -4 & -2 & 1\end{pmatrix}\ \Rightarrow\ |A-\lambda E|=\begin{vmatrix}1-\lambda & 2 & -4\\ 2 & -2-\lambda & 2\\ -4 & -2 & 1-\lambda\end{vmatrix}=0.$$

Bu xarakteristik tenglamaning ildizlarini topamiz:

$$(1-\lambda)^2(-2-\lambda)+16+16-16(-2-\lambda)-4(1-\lambda)-4(1-\lambda)=0$$

$$-(\lambda-1)^2(\lambda+2)+32+32+16\lambda-4+4\lambda-4+4\lambda=0$$

$$(\lambda^2-2\lambda+1)(\lambda+2)-24\lambda-56=0$$

$$\lambda^3+2\lambda^2-2\lambda^2-4\lambda+\lambda+2-24\lambda-56=0$$

$$\lambda^3-27\lambda-54=0$$

$$\lambda_{1,2}=-3;\quad \lambda_3=6$$

Demak, bu kvadratik formaning kanonik ko‘rinishi quyidagicha bo‘ladi:

$$g(y_1,\,y_2,\,y_3)=y_1^2-6y_2^2-9y_3^2$$

Endi topilgan xos sonlar orqali xos vektorlarni aniqlaymiz:

a) $\lambda_1=-3$ bo‘lgan holni qaraymiz:

$$\begin{pmatrix}4 & 2 & -4\\ 2 & 1 & -2\\ -4 & -2 & 4\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}\ \Rightarrow\ \begin{cases}4x_1+2x_2-4x_3=0\\ 2x_1+x_2-2x_3=0\\ -4x_1-2x_2+4x_3=0\end{cases}$$

Bu tenglamalar sistemasining rangi $r=1$ ga, noma'lumlar soni $n=3$ ga teng. Demak, $n-r=2$ ta fundamental yechimlar sistemasi mavjud.

Tenglamalar sistemasidan $x_2=-2x_1+2x_3$ ni topib, fundamental yechimlar sistemasini aniqlaymiz:

$$e_1=\begin{pmatrix}1\\0\end{pmatrix} \text{ va } e_2=\begin{pmatrix}0\\1\end{pmatrix} \Rightarrow F_1=\begin{pmatrix}1\\-2\\0\end{pmatrix} \text{ va } F_2=\begin{pmatrix}0\\2\\1\end{pmatrix}$$

b) $\lambda_2=6$ bo'lgan holni qaraymiz:

$$\begin{pmatrix}-5 & 2 & -4\\ 2 & -8 & -2\\ -4 & -2 & -5\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow \begin{cases}-5x_1+2x_2-4x_3=0\\ 2x_1-8x_2-2x_3=0\\ -4x_1-2x_2-5x_3=0\end{cases}$$

Bu tenglamalar sistemasining rangi $r=2$ ga, noma'lumlar soni $n=3$ ga teng. Demak $n-r=1$ ta fundamental yechimlar sistemasi mavjud.

Tenglamalar sistemasidan $\begin{cases}x_1-22x_2-10x_3=0\\ 2x_2+x_3=0\end{cases}$ ni topib, fundamental yechimlar sistemasini aniqlaymiz:

$$\begin{cases}x_1=2\alpha\\ x_2=\alpha\\ x_3=-2\alpha\end{cases} \Rightarrow F_3=\alpha\begin{pmatrix}2\\1\\-2\end{pmatrix}.$$

Shunday qilib, 3 ta vektordan iborat vektorlar sistemasiga ega bo'ldik:

$$A_1=\begin{pmatrix}1\\-2\\0\end{pmatrix},\quad A_2=\begin{pmatrix}0\\2\\1\end{pmatrix},\quad A_3=\begin{pmatrix}2\\1\\-2\end{pmatrix}$$

Endu bu vektorlar sistemasini ortogonallaymiz (Shmidt formulasi orqali):

$$B_1=\begin{pmatrix}1\\-2\\0\end{pmatrix},$$

$$B_2=A_2-\frac{(B_1,A_2)}{(B_1,B_1)}\cdot B_1=\begin{pmatrix}0\\2\\1\end{pmatrix}-\left(-\frac{4}{5}\right)\cdot\begin{pmatrix}1\\-2\\0\end{pmatrix}=\begin{pmatrix}0,8\\0,4\\1\end{pmatrix},$$

$$B_3=A_3-\frac{(B_1,A_3)}{(B_1,B_1)}\cdot B_1-\frac{(B_2,A_3)}{(B_2,B_2)}\cdot B_2=\begin{pmatrix}2\\1\\-2\end{pmatrix}-0\cdot\begin{pmatrix}1\\-2\\0\end{pmatrix}-0\cdot\begin{pmatrix}0,8\\0,4\\1\end{pmatrix}=\begin{pmatrix}2\\1\\-2\end{pmatrix}.$$

Demak, $B_1=\begin{pmatrix}1\\-2\\0\end{pmatrix}$, $B_2=\begin{pmatrix}0,8\\0,4\\1\end{pmatrix}$, $B_3=\begin{pmatrix}2\\1\\-2\end{pmatrix}$ vektorlar sistemasi ortogonal vektorlar sistemasini hosil qiladi. Endi ortonormallaymiz, ya'ni $\frac{B}{|B|}$ ko'rinishini topamiz:

$$B_1=\begin{pmatrix}1/\sqrt{5}\\-2/\sqrt{5}\\0\end{pmatrix},\quad B_2=\begin{pmatrix}4\sqrt{5}/15\\2\sqrt{5}/15\\\sqrt{5}/3\end{pmatrix},\quad B_3=\begin{pmatrix}2/3\\1/3\\-2/3\end{pmatrix}$$

Bundan

$$\begin{cases}y_1=\frac{1}{\sqrt{5}}x_1+\frac{4\sqrt{5}}{15}x_2+\frac{2}{3}x_3\\ y_2=-\frac{2}{\sqrt{5}}x_1+\frac{2\sqrt{5}}{15}x_2+\frac{1}{3}x_3\\ y_3=\frac{\sqrt{5}}{3}x_2-\frac{2}{3}x_3\end{cases}\Rightarrow\begin{cases}x_1=\frac{1}{\sqrt{5}}y_1-\frac{2}{\sqrt{5}}y_2\\ x_2=\frac{4\sqrt{5}}{15}y_1+\frac{2\sqrt{5}}{15}y_2+\frac{\sqrt{5}}{3}y_3\\ x_3=\frac{2}{3}y_1+\frac{1}{3}y_2-\frac{2}{3}y_3\end{cases}$$

almashtirishni hosil qilamiz.

XULOSA

Kvadratik formalarni o'rganishda ularning kanonik ko'rinishlarini klassifikatsiyaga ajratib o'rganishga to'g'ri keladi. Bunda nafaqat kvadratik formaning kanonik ko'rinishi, balki uni kanonik ko'rinishga keltiruvchi almashtirishlarni ham aniqlash kerak bo'ladi. Shunday ekan har qanday kvadratik formalarni kanonik ko'rinishga keltirish va ularni kanonik ko'rinishga keltiruvchi almashtirishlarni topishning usullarini qo'llay bilish zarurdir.

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